

AUTOMORPHISMS OF INFINITE-DIMENSIONAL HYPERCUBE GRAPH

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ABSTRACT. We consider the infinite-dimensional hypercube graph. This graph is not connected and has isomorphic connected components. We describe the restrictions of its automorphisms to the connected components and the automorphism group of connected component.

1. INTRODUCTION

By [3], typical graphs have no non-trivial automorphisms. On the other hand, the classical Frucht result [4] states that every abstract group can be realized as the automorphism group of some graph (we refer [2] for more information concerning graph automorphisms). In particular, the Coxeter group of type $B_n = C_n$ (the wreath product $S_2 \wr S_n$) is isomorphic to the automorphism group of the n -dimensional hypercube graph H_n .

In this note we consider the infinite-dimensional hypercube graph H_{\aleph_0} . This graph is not connected and has isomorphic connected components. We describe the restrictions of its automorphisms to the connected components (Theorem 1). As a simple consequence, we establish that the automorphism group of connected component is isomorphic to the so-called weak wreath product of S_2 and S_{\aleph_0} (Corollary 1).

2. INFINITE-DIMENSIONAL HYPERCUBE GRAPH

A subset $X \subset \mathbb{Z} \setminus \{0\}$ is said to be *singular* if

$$i \in X \implies -i \notin X.$$

For every natural i each maximal singular subset contains precisely one of the numbers i or $-i$; in other words, if X is a maximal singular subset then the same holds for its complement in $\mathbb{Z} \setminus \{0\}$. Two maximal singular subsets X, Y are called *adjacent* if

$$|X \setminus Y| = |Y \setminus X| = 1.$$

In this case, we have

$$X = (X \cap Y) \cup \{i\} \text{ and } Y = (X \cap Y) \cup \{-i\}$$

for some number $i \in \mathbb{Z} \setminus \{0\}$.

Following Example 2.6 in [8], we say that a permutation s on the set $\mathbb{Z} \setminus \{0\}$ is *symplectic* if

$$s(-i) = -s(i).$$

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A permutation is symplectic if and only if it preserves the family of singular subsets. The group of symplectic permutations is isomorphic to the wreath product $S_2 \wr S_{\aleph_0}$ (we write S_α for the group of permutations on a set of cardinality α , see Section 5 for the definition of wreath product). The action of this group on the family of maximal singular subsets is transitive.

Denote by H_{\aleph_0} the graph whose vertex set is formed by all maximal singular subsets and whose edges are adjacent pairs of such subsets. This graph is not connected. The connected component containing $X \in H_{\aleph_0}$ will be denoted by $H(X)$; it consists of all $Y \in H_{\aleph_0}$ such that

$$|X \setminus Y| = |Y \setminus X| < \infty.$$

Any two connected components $H(X)$ and $H(Y)$ are isomorphic. Indeed, every symplectic permutation s on the set $\mathbb{Z} \setminus \{0\}$ induces an automorphism of H_{\aleph_0} ; this automorphism transfers $H(X)$ to $H(Y)$ if $s(X) = Y$.

Remark 1. It is clear that H_{\aleph_0} can be identified with the graph whose vertices are sequences

$$\{a_n\}_{n \in \mathbb{N}} \text{ with } a_n \in \{0, 1\}$$

and $\{a_n\}_{n \in \mathbb{N}}$ is adjacent with $\{b_n\}_{n \in \mathbb{N}}$ (connected by an edge) if

$$\sum_{n \in \mathbb{N}} |a_n - b_n| = 1.$$

Then one of the connected components is formed by all sequences having a finite number of non-zero elements. The graph H_{\aleph_0} also can be defined as the Cartesian product of infinitely many factors K_2 [6].

3. AUTOMORPHISMS

Every automorphism of H_{\aleph_0} induced by a symplectic permutation will be called *regular*. An easy verification shows that distinct symplectic permutations induce distinct regular automorphisms. Therefore, the group of regular automorphisms is isomorphic to $S_2 \wr S_{\aleph_0}$.

Non-regular automorphisms exist. The following example is a modification of examples given in [1, 9], see also Example 3.14 in [8].

Example 1. Let $A \in H_{\aleph_0}$ and B be a vertex of the connected component $H(A)$ distinct from A . We take any symplectic permutation s transferring A to B . This permutation preserves $H(A)$ and the mapping

$$f(X) := \begin{cases} s(X) & X \in H(A) \\ X & X \in H_{\aleph_0} \setminus H(A) \end{cases}$$

is well-defined. Clearly, f is a non-trivial automorphism of H_{\aleph_0} . Suppose that this automorphism is regular and t is the associated symplectic permutation. For every $i \in \mathbb{Z} \setminus \{0\}$ there exists a singular subset N such that

$$X = N \cup \{i\} \text{ and } Y = N \cup \{-i\}$$

are elements of $H_{\aleph_0} \setminus H(A)$. Then

$$t(N) = t(X \cap Y) = t(X) \cap t(Y) = f(X) \cap f(Y) = X \cap Y = N$$

and

$$N \cup \{i\} = X = f(X) = t(X) = t(N) \cup \{t(i)\} = N \cup \{t(i)\}$$

which implies that $t(i) = i$. Thus t is identity which is impossible. So, the automorphism f is non-regular.

Theorem 1. *The restriction of every automorphism of H_{\aleph_0} to any connected component coincides with the restriction of some regular automorphism to this connected component.*

Remark 2. A similar result was obtained in [9] for the infinite Johnson graph. The proof of that result is based on the same idea, but technically is more complicated.

4. PROOF OF THEOREM 1

Let $A \in H_{\aleph_0}$ and f be the restriction of an automorphism of H_{\aleph_0} to the connected component $H(A)$. For every $X \in H_{\aleph_0}$ we denote by X^\sim the set which contains X and all vertices of H_{\aleph_0} adjacent with X . It is clear that X^\sim is contained in $H(A)$ if $X \in H(A)$.

Lemma 1. *For every $X \in H(A)$ there is a symplectic permutation s_X such that*

$$(1) \quad f(Y) = s_X(Y) \quad \forall Y \in X^\sim.$$

Proof. We can assume that $f(X)$ coincides with X (if $f(X) \neq X$ then we take any symplectic permutation t sending $f(X)$ to X and consider tf). In this case, the restriction of f to X^\sim is a bijective transformation of X^\sim .

For every $i \in \mathbb{Z} \setminus \{0\}$ one of the following possibilities is realized:

- $i \notin X$,
- $i \in X$.

Consider the first case. Then $-i \in X$ and there is unique element of X^\sim containing i , this is

$$(2) \quad Y = \{i\} \cup (X \setminus \{-i\}).$$

Since $f|_{X^\sim}$ is a transformation of X^\sim , $f(Y)$ is adjacent with X and the set $f(Y) \setminus X$ contains only one element. We denote it by $s_X(i)$. It is clear that $s_X(i) \notin X$.

In the second case, $-i \notin X$ and we define $s_X(i)$ as $-s_X(-i)$. Since $s_X(-i)$ does not belong to X , we have $s_X(i) \in X$.

So, s_X is a symplectic permutation on $\mathbb{Z} \setminus \{0\}$ such that

$$s_X(X) = X.$$

Now, we check (1).

Let $Y \in X^\sim$. Then we have (2) for some i and

$$s_X(Y) = \{s_X(i)\} \cup (s_X(X) \setminus \{-s_X(i)\}) = \{s_X(i)\} \cup (X \setminus \{-s_X(i)\})$$

is unique element of X^\sim containing $s_X(i)$. On the other hand, $s_X(i)$ belongs to $f(Y)$ by the definition of s_X . Therefore, $f(Y)$ coincides with $s_X(Y)$. \square

Lemma 2. *If $X, Y \in H(A)$ are adjacent then $s_X = s_Y$.*

Proof. Since X, Y are adjacent, we have

$$X = \{i\} \cup (X \cap Y) \quad \text{and} \quad Y = \{-i\} \cup (X \cap Y)$$

for some $i \in X$. We can assume that

$$f(X) = X \quad \text{and} \quad f(Y) = Y.$$

Indeed, in the general case

$$f(X) = \{j\} \cup (f(X) \cap f(Y)) \text{ and } f(Y) = \{-j\} \cup (f(X) \cap f(Y))$$

(since $f(X)$ and $f(Y)$ are adjacent); we take any symplectic permutation t sending j and $f(X) \cap f(Y)$ to i and $X \cap Y$ (respectively) and consider tf .

Then

$$s_X(X \cap Y) = s_X(X) \cap s_X(Y) = f(X) \cap f(Y) = X \cap Y;$$

similarly,

$$s_Y(X \cap Y) = X \cap Y.$$

We have

$$(X \cap Y) \cup \{i\} = X = f(X) = s_X(X) = s_X((X \cap Y) \cup \{i\}) = (X \cap Y) \cup \{s_X(i)\}$$

and the same arguments show that

$$(X \cap Y) \cup \{i\} = (X \cap Y) \cup \{s_Y(i)\}.$$

Therefore,

$$s_X(i) = s_Y(i) = i \text{ and } s_X(-i) = s_Y(-i) = -i.$$

Now, we show that the equality

$$(3) \quad s_X(j) = s_Y(j)$$

holds for every $j \neq \pm i$. Since s_X and s_Y are symplectic, it is sufficient to establish (3) only in the case when $j \notin X \cup Y$. Indeed, if $j \in X \cap Y$ then $-j$ does not belong to $X \cup Y$.

Let j be an element of $\mathbb{Z} \setminus \{0\}$ which does not belong to $X \cup Y$. Then $-j \in X \cap Y$ and

$$X' := \{j\} \cup (X \setminus \{-j\}) \in X^\sim, \quad Y' := \{j\} \cup (Y \setminus \{-j\}) \in Y^\sim$$

are adjacent. Hence

$$f(X') = s_X(X') = \{s_X(j)\} \cup (X \setminus \{-s_X(j)\})$$

and

$$f(Y') = s_Y(Y') = \{s_Y(j)\} \cup (Y \setminus \{-s_Y(j)\})$$

are adjacent. The latter is possible only in the case when $s_X(j) = s_Y(j)$. \square

Using the connectedness of $H(A)$ and Lemma 2, we establish that $s_X = s_Y$ for all $X, Y \in H(A)$.

5. AUTOMORPHISMS OF CONNECTED COMPONENTS

Let G_1 and G_2 be permutation groups on sets X_1 and X_2 , respectively. Recall that the *wreath product* $G_1 \wr G_2$ is a permutation group on $X_1 \times X_2$ and its elements are compositions of the following two types of permutations:

- (1) for each element $g \in G_2$, the permutation $(x_1, x_2) \rightarrow (x_1, g(x_2))$;
- (2) for each function $i : X_2 \rightarrow G_1$, the permutation $(x_1, x_2) \rightarrow (i(x_2)x_1, x_2)$.

Consider the subgroup of $G_1 \wr G_2$ whose elements are compositions of all permutations of type (1) and permutations of type (2) such that the set

$$\{x_2 \in X_2 : i(x_2) \neq \text{id}_{X_1}\}$$

is finite. This is a proper subgroup only in the case when X_2 is infinite; it will be called the *weak wreath product* and denoted by $G_1 \wr_w G_2$.

Corollary 1. *The automorphism group of connected component of H_{\aleph_0} is isomorphic to the weak wreath product $S_2 \wr_w S_{\aleph_0}$.*

Proof. Let $A \in H_{\aleph_0}$ and f be an automorphism of the connected component $H(A)$. By the previous section, f is induced by a symplectic permutation s . Since $f(A) = s(A)$ belongs to $H(A)$, the set $s(A) \setminus A$ is finite. So, the automorphism group of $H(A)$ is isomorphic to the group of symplectic permutations s such that the set $s(A) \setminus A$ is finite. The latter group is isomorphic to the weak wreath product $S_2 \wr_w S_{\aleph_0}$ (indeed, we can identify the set $\mathbb{Z} \setminus \{0\}$ with the Cartesian product $\mathbb{Z}_2 \times A$ and the group S_{\aleph_0} with the group of all permutation on A). \square

Remark 3. The latter result is not new. Since H_{\aleph_0} is the Cartesian product of infinitely many factors K_2 , Corollary 1 can be drawn from the well-known results concerning the automorphism group of Cartesian product of graphs [5, 7].

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